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Linear sigma model toolshed for D-brane physics

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Abstract: Building on earlier work [1], we construct linear sigma models for strings on curved spaces in the presence of branes. Our models include an extremely general class of brane-worldvolume gauge field configurations. We explain in an accessible manner the mathematical ideas which suggest appropriate worldsheet interactions for generating a given open string background. This construction provides an explanation for the appearance of the derived category in D-brane physics complementary to that of [2].

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1. Introduction

Main Entry: de · noue · ment
Function: noun
1. the final outcome of the main dramatic complication in a literary work
2. the resolution of a complex sequence of events

The study of supersymmetric D-branes in curved spaces is a dual-purpose endeavor. On one hand, these objects provide new probes of the stringy physics which was uncovered principally by the application of mirror symmetry. On the other hand, when coupled with orientifolds, they provide a large, relatively uncharted class of quasi-realistic string vacua.

Thus far, only a small part of the spectrum of branes on Calabi-Yau (CY) manifolds has been studied. While a classification of these objects, even in the geometrical limit, is lacking, an avenue toward systematic exploration is provided by the fact that all bundles have a description in terms of simpler components. Such a description, called a "resolution" (see [38] or [39, Chapter II, Corollary 5.18]) is a sequence of maps between sums of rank-one bundles which encodes the non-triviality of the bundle of interest. As a means of clearly laying out the space of D-brane states, these sequences appear promising [24]. It should be possible to use this sequence data to make a useful model of the stringy physics.

A tractable description of the conformal field theory (CFT) describing the string dynamics on a CY is lacking (for any branes, or even in their absence) except for special values of the closed string moduli. A useful model of much of the physics at all points in moduli space is provided by linear sigma models (LSM's) [5]. Such models are two-dimensional quantum field theories which approach the desired CFT at long distances. Linear sigma models are in many cases the only available tools for extracting information about string backgrounds in the small-volume region, including the persistence of such backgrounds beyond perturbation theory. The LSM is the ideal framework to

Harness the awesome power of homological algebra. Already the shortest such sequences ("monads") have appeared in linear sigma model constructions, first for heterotic strings [5, 29], and more recently for open strings [10].

In this paper, we make linear models for bundles whose resolution has an arbitrary number of nodes.[1] More generally, the models we make generate any bundle which can be realized as the cohomology of a sequence of sums of line bundles.

The sequences which encode the data for these models are the same ones that appear as building-blocks for the derived category of [24].[2] As in other incarnations of open string field theory [40], the condensation of spacetime tachyons is manifested on the string worldsheet as flow to the infrared. In our construction, the equivalence of quasi-isomorphic complexes is a consequence of universality in the sense of the renormalization group.

There a second way in which our work relates to [24] which we explain in the concluding section.

In addition to any motivation from recent exciting ideas about abstract descriptions of D-brane spectra, our construction would be needed for an attempt to make contact with phenomenology through this type of four-dimensional $\mathcal{N} = 1$ string vacuum. Certainly any systematic exploration of this class of vacua would incorporate the more general models we construct.

The plan of the paper is as follows. In section 2 we (p)review the construction [10] of a linear sigma model coupling open strings to B-type branes, and more specifically to a bundle defined by a one-step sequence. In section 3 we explain why this construction fails for more general bundles whose resolutions do not terminate

[1] It was suggested recently [24] that this would be useful.

[2] Early work in this direction includes [44, 42, 43].

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after one step, and explain our solution. In section 4 we discuss a linear model for an even broader class of bundles, namely those which are not pullbacks of bundles on the ambient space in which the CY is embedded. In section 5 we explore the relation of our linear models to possible linear models for heterotic strings whose left-moving fermions couple to these more general bundles. We close with a discussion of applications and the extension of this set of ideas to CY's and branes which are not complete intersections. In appendix A we give more details about our models. In appendix B we describe another formulation of our models, of which the one in the body of the paper is a gauge-fixed version.

Other work on linear models for open strings includes [11]–[15].

2. Brief review of the construction for one step

We present this discussion in the context of a CY hypersurface in projective space, but the generalization to any complete intersection in a toric variety should be clear. The simplest example of the monad construction defines a holomorphic vector bundle, V, from the sequence of maps

$$0 \to V \to E_1 \equiv \bigoplus_{a_1} \mathcal{O}(n_{a_1}) \xrightarrow{d_1} E_2 \equiv \bigoplus_{a_2} \mathcal{O}(n_{a_2}) \to 0\,.$$

The first map is inclusion and the second map is

$$d_1 : E_1 \to E_2 \beta^{a_1} \mapsto \sum d_1{}^{a_2}{}_{a_1}(\phi)\beta^{a_1}\,.$$

This sequence is *exact* in the sense that the kernel of one map is the image of the previous one. For our purposes, the bundles E_l are sums of powers of the hyperplane bundle over a projective space. We want to consider type II string theory on a Calabi-Yau hypersurface, X, in the projective space. A linear sigma model which realizes open strings ending on a brane wrapping X with gauge bundle V works as follows [10].

The bulk fields are the same as in the $(2,2)$ linear model for X. For ease of exposition we present the model for a hypersurface in $\mathbb{P}^4$ defined by a homogeneous polynomial G of degree $d = 5$. Then one has a single $U(1)$ gauge multiplet under which the chiral multiplets Φ^i, $i = 1 \dots 5$ carry charge 1 and P carries charge -5. There is a (2,2) superpotential $W_{bulk} = PG(\Phi)$. In accordance with the notation of the relevant (B-type) topological field theory, we refer to the fermions in a bulk chiral multiplet as

$$\Theta = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)\,, \qquad \eta = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)\,.$$

We refer to [5] for more details.

We give these fields couplings to boundary matter which respect B-type supersymmetry. By B-type supersymmetry we mean a subalgebra of the bulk (2,2) supersymmetry generated by a linear combination of Q_+ and Q_- rather than a linear combination of Q_+ and $Q_-^\dagger$. Boundary conditions preserving such a subalgebra are associated with branes of even codimension in the CY [20]. We call the conserved supercharges Q and $Q^\dagger$. The representation theory of their algebra is similar to that of (0,2) two-dimensional supersymmetry and is worked out in [10]. The two kinds of multiplets we will need are: Fermi multiplets, β, have a fermion as their lowest component, and chiral multiplets, $\wp$, have a boson as their lowest component. Both satisfy either the chiral constraint, $Q^\dagger(\beta, \wp) = 0$, or some deformation thereof. The consistency conditions on such deformations will play a key role in building the more general bundles.

2.1 About kinetic terms

A Fermi multiplet contains a complex fermion, β, and an auxiliary complex boson, b. A supersymmetric kinetic term is [3]

$$\int d^2\theta \beta\dagger\beta = b\dagger b - i(\nabla_0\beta\dagger)\beta + i\beta\dagger(\nabla_0\beta)\,,$$

where

$$\nabla_0 \equiv \frac{i}{2}\{Q, Q\dagger\}\,.$$

A boundary chiral multiplet contains a complex boson, $\wp$, and a fermion ξ. In addition to a usual kinetic term of the form

$$\int d^2\theta \wp^\dagger\nabla_0\wp\,,$$

we add a magnetic field term:

$$\int d^2\theta B\wp^\dagger\wp = B\xi^\dagger\xi + B\wp^\dagger\nabla_0\wp - (\nabla_0 B\wp^\dagger)\wp\,,$$

where B is a constant. Consider a limit where B is big so that we can ignore the kinetic terms for $\wp$, c.f. e.g.[41]. This approximation can be justified by the fact that this term is less irrelevant than the kinetic one. In that case, the momentum conjugate to $\wp$ is $iB\wp^\dagger$. So canonical quantization gives

$$[\wp, \wp^\dagger] = 1/B\,.$$

The coupling to the magnetic field masses up the fermion ξ and it halves the number of $\wp$ degrees of freedom — i.e. the set of states made by $\wp$ is now just the Hilbert space of a harmonic oscillator. It makes the $\wp$ multiplet into an exact bosonic analog of a Fermi multiplet. Since we work in an approximation where the magnetic field term dominates the kinetic term, we will drop the usual kinetic term altogether and adopt a normalization for $\wp$ such that $B \equiv 1$.

[3] We use gauge-covariantized supercharges, so we omit the usual e^{qV}'s.

2.2 The model for a monad

On the boundary of the string, introduce Fermi multiplets, β^{a_1}, which live in the bundle E_1, and chiral multiplets, $\wp_{a_2}$, which live in the bundle $E_2^\star$. In this case the half-superspace integrand which generates the bundle V is

$$W = \wp_{a_2} d_1{}^{a_2}{}_{a_1} \beta^{a_1}$$

which we abbreviate as

$$W = \wp d_1 \beta\,.$$

(We will suppress indices whenever possible!)

After integrating out the auxiliary bosons in β_a, this leads to the following interactions on the boundary

$$\mathcal{L} = \cdots + \xi_{a_2} d_1{}^{a_2}{}_{a_1} \beta^{a_1} + h.c. + \sum_{a_2} |\wp_{a_2} d_1{}^{a_2}{}_{a_1}|^2\,.$$

The finishing touch we need to put on the monad model is to implement a charge projection on the boundary which guarantees the right number of Chan-Paton (CP) states — i.e., that only one Chan-Paton fermion at a time will be excited, no more and no less. Specifically, we gauge the symmetry which acts only on boundary fields, as

$$\wp \mapsto e^{i\gamma}\wp \beta \mapsto e^{-i\gamma}\beta\,.$$

This is done by adding a supersymmetry singlet, one-dimensional gauge field a_0 on the boundary, which couples as

$$\mathcal{L}_a = \int_{\partial\Sigma} a_0 (j_S - 1)\,,$$

where j_S is the boundary symmetry charge. Since each excitation of the β fermions is in effect a 'quark' at the left endpoint of the open string (or an 'antiquark' at the right endpoint), the charge projection restricts the system to a subsector of the Hilbert space where the open string worldsheet couples to the gauge field as a section of $V \otimes V^*$ [10].

3. Any resolution

Consider a bundle V over a Calabi-Yau manifold which is a hypersurface in projective space. Such a bundle has a resolution of the form

$$0 \to V \to E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} E_3 \to \cdots \tag{3.1}$$

which is an exact sequence and the E_ls are direct sums of line bundles. The map from E_l to E_{l+1} depends on ϕ and we will call it d_l. Let k_l be the rank of E_l.

At this point, the important point to make is what goes wrong with the naive model when d_1 has a cokernel. This is twofold: firstly, there are extra massless $\wp$s on the boundary, each of which gives an energetically degenerate harmonic-oscillator spectrum of states, leading to the wrong spectrum of CP factors. Secondly, and perhaps more importantly, a *generic* deformation of the matrix elements of d_1 will destroy the brane. This happens because the matrix d_1 imposes overconstrained conditions on β, which are inconsistent rather than redundant for a generic deformation. So under a random deformation, the equations of motion for the $\wp$s and βs will simply set them to zero at low energies, meaning that no states will satisfy the boundary charge projection — there will be no CP factors. This is connected with the problem of massless $\wp$'s in that the superpartners of the $\wp$'s are the Goldstone fermions for the spontaneous breaking of supersymmetry.

Now for a model with the correct behavior. The sections of the line bundles which we introduce will be alternately Fermi multiplets, $\beta^a_{(l)}$, and chiral multiplets, $\wp_{a(l+1)}$. They are Fermi multiplets at the first step, as in the monad case, because it is states of massless fermions that play the role of the CP factors (or the left-moving current algebra in a (0,2) theory). The basic idea is that the fermions at the third step pair up with the "extra" massless fermion partners of the $\wp$'s at the second step (which arise because d_1 has a cokernel). If the sequence does not terminate there, the partners of the bosons at the fourth step pair up with the extra fermions at the third step, and this process of lifting kernels and images continues until the sequence terminates.

In the general case the boundary symmetry acts on all chiral multiplets with one phase and all Fermi multiplets with the opposite phase.

For ease of discussion, we give the details for a bundle with a two-step resolution:

$$0 \to V \to E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} E_3 \to 0\,. \tag{3.2}$$

We will also assume for simplicity that the line bundles making up E_i at each step are the same, i.e.,

$$E_i \equiv \mathcal{O}(n_i)^{\oplus k_i}\,.$$

This assumption is in no way essential to the construction.

Then the degrees $\Delta_{1,2}$ of the polynomials defining the maps $d_{1,2}$ are $n_2 - n_1$ and $n_3 - n_2$, respectively. Here is the field content:

- Bulk fields: the usual (2,2) multiplets (ϕ, P_{bulk}, σ) where P_{bulk} is the *bulk P*-multiplet.
- A set of boundary fermi multiplets β^i, a smoothly varying subspace of which will define the desired bundle. They satisfy the usual chiral constraint $\{Q\dagger, \beta\} = 0$ and their gauge charge is $n \equiv n_1$.
- A set of boundary chiral multiplets $\wp_i$ with gauge charges $-n_2 = -(n + \Delta_1)$. Instead of the usual chiral constraint $[Q\dagger, \wp] = 0$ these will obey a deformed chiral constraint, which will fix the $Q\dagger$ variation of $\wp$ in terms of the other fields of the problem.

• A set of boundary multiplets β' which boundary-symmetry charge $+1$ and gauge charge $n_3 \equiv n + \Delta_1 + \Delta_2$. They will turn out to be forced to satisfy the *opposite* of the chiral constraint: $\{Q, \beta'\} = 0$.

Now for the interactions. The half-superspace integral will be:

$$\int d\theta\, W \equiv \int d\theta\, \wp d_1 \beta\,.$$

We can add the obvious gauge-invariant kinetic term for the β, β' multiplets:

$$\int d^2\theta \cdot \beta\dagger\beta + \beta'^{\dagger}\beta'\,.$$

The key to obtaining the correct physics is the multiplet structure of $\wp$:

Since d_1 has a cokernel, the condition for the superpotential W to be annihilated by $Q\dagger$ does not require $Q\dagger$ to annihilate $\wp$. Indeed, $[Q\dagger, \wp] = (anything) \cdot d_2$ will suffice.

There is an obvious choice: $[Q\dagger, \wp] = i\beta'^{\dagger} d_2$. Since for a two-step sequence d_2 is onto, the nilpotence of $Q\dagger$ forces $\{Q\dagger, \beta'^{\dagger}\} = 0$ — so β' satisfies the opposite of the usual chiral constraint.

This constraint completely determines the supersymmetry transformations of the new multiplet. This is one of the two key ingredients in this new type of model. The second is the observation that for a boson living on the boundary, it is consistent to add to the lagrangian a magnetic field coupling for the complex target space fiber coordinates:

$$\int d^2\theta\, \wp\wp^{\dagger}\,. \tag{3.3}$$

In fact it is necessary to add this term to obtain the mass terms and Yukawa couplings that give the desired physics in the infrared.

Rather than listing all the terms in the action and the supersymmetry transformations here, we leave that to the Appendix. In this section we will discuss only the terms which will be important for causing the model to flow to the correct brane configuration at low energies, in the 'large radius' phase of the worldsheet theory.

To clarify the content of the low energy theory we begin by integrating out the auxiliary fields b in the fermi multiplets and also the superpartners ξ of $\wp$, which become auxiliary at low energies in the presence of the large magnetic field. With auxiliary fields eliminated, the key terms in the lagrangian are then:

• Derivative terms for the physical bosons and fermions of the system:

$$i \cdot \left[\beta\dagger(\nabla_0\beta) + (\nabla_0\wp)\wp\dagger + \beta'^{\dagger}(\nabla_0\beta') - h.c.\right]\,.$$

For the fermions these are just standard kinetic terms; for the bosons the standard kinetic term is irrelevant and their dynamics at this scale is dominated by lowest Landau level physics; the number of physical degrees of freedom is effectively reduced by a factor of two and the complex bosonic fiber becomes a product of noncommutative $\mathbb{C}^1$'s. Now the boson $\wp$ really does have similar kinematics to the fermions β and β', except with opposite statistics, i.e., $[\wp_i, \wp\dagger_j] = \delta_{ij}$.

- Mass terms for the fermions:

$$\beta\dagger(d_1\dagger d_1)\beta + \beta'^{\dagger}(d_2 d\dagger_2)\beta' \,. \tag{3.4}$$

- Mass terms for the bosons:

$$\wp(d_1 d\dagger_1 + d\dagger_2 d_2)\wp\dagger \,. \tag{3.5}$$

Note that because of the large magnetic field, the physical mass of the bosons is proportional to the coefficient of $|\wp|^2$ rather than to the square root of that coefficient.

In the next subsection we explain why these potentials and Yukawa couplings are preicisely what we need to yield the correct sigma model at low energies.

Interestingly, the supersymmetry transformations take a very nice form with ξ and the other auxiliary fields integrated out:

$$\begin{array}{rlrl} \{Q,\beta\} &= -id\dagger_1\wp\dagger & \{Q,\beta\dagger\} &= 0 \\ \{Q\dagger,\beta\} &= 0 & \{Q\dagger,\beta\dagger\} &= +i\wp d_1 \\ [Q,\wp\dagger] &= +id\dagger_2\beta' & [Q,\wp] &= -i\beta\dagger d\dagger_1 \\ [Q\dagger,\wp\dagger] &= +id_1\beta & [Q\dagger,\wp] &= -i\beta'^{\dagger}d_2 \\ \{Q,\beta'\} &= 0 & \{Q,\beta'^{\dagger}\} &= -i\wp d\dagger_2 \\ \{Q\dagger,\beta'\} &= +id_2\wp^{\dagger} & \{Q\dagger,\beta'^{\dagger}\} &= 0\,. \end{array} \tag{3.6}$$

We note again that these transformations are consistent with the supersymmetry algebra (in particular, $Q^2 = 0$) because $d_2 \circ d_1 = 0$. The relation $\{Q, Q^{\dagger}\} = 2H$ holds on-shell, that is when the fields satisfy their (first-order) equations of motion. Although the limit in which the two-derivative kinetic term for the bosons vanishes simplifies the algebra greatly, the model is consistent without using this approximation.

The generalization to sequences of an arbitrary number of steps should be clear. The $Q\dagger$-variation of a field associated with a given node encodes the previous map, while the Q-variation encodes the next map. So, for example, to make a 3-step sequence, we would add some $\wp'$ fields satisfying the undeformed chiral constraint $[Q\dagger, \wp'] = 0$ and deform the constraint on β' to $\{Q, \beta'\} = d_3\dagger\wp'^{\dagger}$. The model works for a sequence of arbitrary length; one never needs to add another superpotential term, only deform the chiral constraint by hand at each step according to the data of the sequence and add the appropriate full-superspace terms. In the appendix we write down the model for a sequence of arbitrary length.

3.1 Large-radius analysis

The point of this section is to prove that we get the right low-energy behavior for the two-step model from the interactions discussed in the previous subsection. Make the bulk FI coefficient r large and positive, so that we are in the large-radius CY phase of the bulk theory.

Let us determine the supersymmetric vacuum of the theory by setting to zero the supersymmetry variations in equation (3.6) :

$$\begin{aligned} 0 &= d_1\beta \\ 0 &= \wp d_1 \\ 0 &= \wp d_2^\dagger \\ 0 &= d_2^\dagger \beta' \,. \end{aligned} \tag{3.7}$$

The first equation tells us that the massless βs live in the kernel of d_1. Since d_2 is surjective, the last equation tells us that β' must vanish. The two middle equations tell us that $\wp$ is closed and co-closed which, since the sequence is exact at E_2, tells us that $\wp$ must vanish as well. This is the desired physics.

We would arrive at the same conclusion by a direct examination of the lagrangian, without making use of supersymmetry. The statement that the sequence we examine has no cohomology at the middle node is the statement that there is no nonzero fiber annihilated both by d_2 and by $d_1^\dagger$. As a result, $d_2^\dagger d_2 + d_1 d_1^\dagger$ is an invertible matrix, so all components of $\wp$ are set to zero at low energies by their equations of motion. Similar reasoning shows that all β' are set to zero by their mass terms at low energies, and that the surviving subset of the β's is the kernel of d_1, just as we wanted.

Note that this analysis makes it clear that there is a direct relation between cohomology of the sequence and massless worldsheet fields. In particular, if under a deformation the defining polynomials the sequence fails to be exact at some point, $\phi_\star$, in the CY, closed and co-closed will no longer imply zero and we will find massless β's and $\wp$'s.

In summary, as in the simpler case, given a non-degeneracy condition for the sequence (analogous to $G = d_1 = 0$ has no solutions in the monad case [29]) all of the $\wp$'s vanish in vacuum, and the mass matrix for the fermions imposes the sequence.

3.2 Examples

The first example we study is a very trivial one, namely a multi-step resolution for a twobrane on a two-torus with a trivial line bundle. Make the T^2 as a cubic hypersurface in $\mathbb{P}^2$. Consider the Koszul complex over the ambient $\mathbb{P}^2$,

$$0 \to V \overset{i}{\hookrightarrow} \mathcal{O}(1)^{\oplus 3} \xrightarrow{d_1} \mathcal{O}(2)^{\oplus 3} \xrightarrow{d_2} \mathcal{O}(3) \to 0\,,$$

where the maps are

$$d_1 = \begin{pmatrix} 0 & \phi_2 & -\phi_1 \\ -\phi_2 & 0 & \phi_0 \\ \phi_1 & -\phi_0 & 0 \end{pmatrix} \quad \text{and} \quad d_2 = (\phi_0, \phi_1, \phi_2)\,;$$

the inclusion is induced by

$$i = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix}.$$

One can see by computing its Chern classes that the line bundle V defined by this sequence is in fact trivial.[4]

To model a string coupling to this brane, we add on its boundary three fermi fields β of charge 1, three chiral fields $\wp$ of charge -2, and another fermi field β' of charge 3. We add the superpotential

$$W = \wp d_1 \beta = \epsilon_{ijk}\beta^i \wp^j \phi^k$$

and implement the chiral constraints and charge projections discussed above.

Obviously this brane could also be constructed by simply adding a neutral fermion on the end of the string. To see the relation to the above model, define an effective neutral fermion, γ, by

$$\beta^i = \phi^i \gamma + \text{massive}\,.$$

Setting to zero $\wp, \beta'$ and the massive components of β then solves the vacuum equations identically. So the multistep model does flow in the IR to the theory of a single neutral fermion.

For a less trivial example, we study the pullback to the CY of the tensor square $V \equiv \mathrm{T}^*\mathbb{P}^n \otimes \mathrm{T}^*\mathbb{P}^n$ of the cotangent bundle of $\mathbb{P}^n$. This is defined by the sequence

$$0 \to V \overset{i}{\hookrightarrow} \mathcal{O}(-2)^{\oplus(n+1)^2} \xrightarrow{d_1} \mathcal{O}(-1)^{\oplus 2n+2} \xrightarrow{d_2} \mathcal{O}(0) \to 0\,,$$

where the maps are

$$d_1 : \beta_{ij} \mapsto \left(\phi^j \beta_{ij}, \phi^i \beta_{ij}\right) \quad \text{and} \quad d_2 : (\wp, \hat{\wp}) \mapsto \phi^i(\wp_i - \hat{\wp}_i)\,.$$

To build this bundle, we add $(n+1)^2$ fermi multiplets β_{ij} of charge -2, $2n+2$ chiral multiplets $\wp_i$ and $\hat{\wp}_i$ of charge -1, and one neutral fermi multiplet β'. The superpotential is

$$W = \wp_i \phi^j \beta_{ij} + \hat{\wp}_j \phi^i \beta_{ij}$$

and the nontrivial deformed constraints are

$$\begin{aligned} [Q, \wp_i] &= -i\beta_{ij}\dagger\phi^{j\dagger} & [Q, \hat{\wp}_j] &= -i\beta_{ij}\dagger\phi^{i\dagger} \\ [Q\dagger, \wp_i] &= -i\beta'^{\dagger}\phi^i & [Q\dagger, \hat{\wp}_i] &= i\beta'^{\dagger}\phi^i \\ \{Q, \beta'^{\dagger}\} &= -i(\wp - \hat{\wp})\dagger\phi^{i\dagger}\,. \end{aligned} \tag{3.8}$$

We have given two examples of smooth bundles with constant fiber dimension using resolutions of finite length. It would be nice to understand more about the physics of the sheaves defined by more general choices of ranks and charges in a multistep sequence (3.1).

[4] This statement is false. The example of the CY 1-fold is special in that the target space is not simply connected; however the existence of a global section (demonstrated below) means that the bundle is in fact trivial.

4. Bundles which do not extend to the ambient space

In fact we lose some generality by considering only pullbacks of bundles on the ambient toric variety. Bundles which extend generate in general a sublattice of finite index in the lattice of all K-theory classes of bundles on the target variety.

In order to see how to generate linear models for more general bundles, let us consider the case of a hypersurface defined by $G(\phi) = 0$. The bulk then has a single P-field and a superpotential $W_{\text{bulk}} \equiv PG(\phi)$. Non-extending bundles can be realized as the cohomology of a sequence *over the coordinate ring of the variety*, i.e. the cohomology of a set of maps d_n such that $d_{n+1}d_n = M_{n+1|n}G(\phi)$ for some matrix $M_{n+1|n}$ of holomorphic polynomials.

The key fact in the following construction is that

$$d_{n+2}M_{n+1|n} = M_{n+2|n+1}d_n \,.$$

Proof: $G(\phi)(d_{n+2}M_{n+1|n} - M_{n+2|n+1}d_n) = (d_{n+2}d_{n+1})d_n - d_{n+2}(d_{n+1}d_n) = 0$ by associativity. Since G is nonvanishing and polynomial rings contain no zero divisors, that means the second factor $d_{n+2}M_{n+1|n} - M_{n+2|n+1}d_n$ must vanish and the statement follows.

We now set the bulk η multiplet on shell; in particular this means $F^{\dagger^P} \equiv \{Q\dagger, \eta^{\dagger^P}\} = G(\phi)$. For a two-step resolution we take the boundary superpotential to be

$$W = (\wp d_1 - \eta^{\dagger^P}\beta'^{\dagger}M_{2|1})\beta$$

and the (deformed) chiral constraints to be

$$\{Q\dagger, \beta\} = 0 \qquad [Q\dagger, \wp] = \beta'^{\dagger}d_2 \qquad \{Q\dagger, \beta'\} = id_2\wp\dagger \,.$$

Clearly the supersymmetry algebra closes and the superpotential is annihilated by $Q\dagger$.

The extension to a multistep sequence over a hypersurface is straightforward. For three steps, for instance, the superpotential is the same, and the supersymmetry transformations are

$$\begin{aligned}
\{Q\dagger, \beta\} &= 0 \\
[Q\dagger, \wp] &= \beta'^{\dagger}d_2 - \eta^{\dagger^P}\wp' M_{3|2} \\
\{Q\dagger, \beta'\} &= id_2\wp\dagger \qquad \{Q\dagger, \beta'^{\dagger}\} = \wp' d_3 \\
[Q\dagger, \wp'] &= 0 \,.
\end{aligned} \tag{4.1}$$

For an arbitrary number of steps, take the same superpotential and use the constraints

$$
\begin{aligned}
\{Q\dagger, \beta\} &= 0 \\
[Q\dagger, \wp] &= -i\beta'^{\dagger} d_2 - \eta^{\dagger^P} \wp' M_{3|2} \\
\{Q\dagger, \beta'^{\dagger}\} &= i\wp' d_3 - \eta^{\dagger^P} \beta''^{\dagger} M_{4|3} \\
&\cdots
\end{aligned}
\tag{4.2}
$$

5. Discussion

5.1 Relation to heterotic models

One might hope that one could use similar technology to make linear models for heterotic strings coupling to these bundles. In fact, our spectrum of fields was motivated by the anomaly coefficients and cancellation conditions that one would have in the heterotic case. In the table at left we write the gauge charges, boundary symmetry charges, and R-symmetry charges of our fields for a generic two-step model.

Field	q_G	q_L	q_R
Φ_i	w_i	0	$\frac{w_i}{m}$
Γ	$-d$	0	$1-\frac{d}{m}$
β	n_1	1	$\frac{n_1}{m}$
$\wp$	$-n_2$	-1	0
β'	n_3	1	$1-\frac{d_2}{m}$

Table 1: The gauge charges and "left-moving U(1)" charges of the fields.

Now imagine that they are instead representations of (0,2) supersymmetry, as in [29]. We call the boundary symmetry charge q_L because in its heterotic incarnation, it is the charge under the left-moving U(1) which becomes part of the spacetime gauge group (and a Z_2 subgroup of which provides one of the GSO projections). Using these charges we would calculate the anomaly in the left-moving $U(1)$ to be

$$\mathcal{A}(L,G) \propto \sum_{fields} q_G q_L = k_1 n_1 - k_2 n_2 + k_3 n_3\,.$$

This is just $c_1(V)$.[5] Note that consistency only requires this to vanish mod 2, since only a Z_2 subgroup of $U(1)_L$ is gauged. The gauge anomaly would be

$$\mathcal{A}(G,G) \propto \sum_{fields} (-1)^{fermi} q_G q_G \propto c_2(V) - c_2(X)\,.$$

[5] The chern classes of the bundle V are determined by the sequence (3.1) to be

$$c(V) = \prod_{l=0} c(E_l)^{(-1)^l}$$

and in particular $c_1(V) = \sum_l (-1)^l \sum_{a_l} n_{a_l} J$, and $c_2(V) = \sum_l \sum_{a_l \neq a'_l} n_{a_l} n_{a'_l} J^2$ where J is the $(1,1)$ form on the CY.

The anomaly in the $\mathrm{U}(1)_R$ symmetry would be (modulo the gauge anomaly)

$$\mathcal{A}(R,G) \propto \sum_{fields} q_G q_R = \sum_i w_i - d\,,$$

the first chern class of the hypersurface.

As in [29, 30] we can calculate the left-moving central charge of our model as an anomaly-matching coefficient in the massive theory. Specifically, the left-moving central charge is the would-be (if it were gauged) quadratic anomaly of the left-moving $U(1)$ symmetry. This is $r = k_0 - k_1 + k_2$, the rank of the bundle, as expected.

However, while the spectrum seems to be correct, the interactions we would add to give the right vacuum structure do not respect two-dimensional Lorentz invariance. An equivalent way to phrase the obstruction is that the desired interactions break the $U(1)$ R-symmetry which is thought to flow to the R-symmetry of the right-moving $\mathcal{N} = 2$ superconformal algebra. It is possible that a new R-symmetry appears in the IR, but without a UV candidate for the IR R-symmetry one loses some confidence that one has identified the correct LSM. This is puzzling, but we still hope to find a LSM for the multi-step resolution in the heterotic case. Given that nonperturbative conformal invariance of (0,2) models has only been proven using the linear models [32], it would be fascinating if no heterotic LSM could be found for multi-step bundles.

We also hope to find an argument that motivates the field content directly in the open string case, such as a direct relation between the RR charge of the D-brane configuration and the spectrum of worldsheet fields.

5.2 Other phases and singularities

One of the great successes of the linear sigma model approach to CY physics is an automatic description of the stringy physics of small-volume phases of the theory. We are still working out the behaviour of the monad theories when the FI parameter is large and negative [10], and we leave the application of that analysis to these more intricate models for future work.

But there is a possibility of new physics from our multistep bundles. When the moduli of the bundle are deformed in such a way that the maps beyond the first step degenerate, cohomology will appear at higher nodes. It would be interesting to see if the resulting massless worldsheet fields have any special signatures in the spacetime physics.

We have been assuming thoughout this paper that the sequence

$$0 \to E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} E_3 \to \cdots \tag{5.1}$$

was chosen so that its cohomology was a bundle. In fact, the cohomology of such a sequence will generically be a more general coherent sheaf (espoused by Harvey and Moore [21] as the proper mathematical characterization of wrapped D-branes). Work is in progress to harness this fact to study D-brane configurations without space-filling branes [10].

5.3 Other classes of models

In addition to the heterotic models discussed above, we are hoping to extend the construction to:

1. (2,2) linear models for varieties which are not complete intersections. The generic CY manifold is such a beast. If a variety is not a complete intersection, it means that the number of defining equations is bigger than its codimension. As a result, there are relations among these equations, and in general relations among these relations ... There is again a sequence of maps resolving the ideal of the variety. We have made some progress towards such (2,2) models using extra gauge symmetries.

2. Open strings in the presence of branes wrapping submanifolds which are not complete intersections.

If we can accomplish item 1 above, our technology will be very useful for the program of [24]. In particular, the 4d $\mathcal{N} = 1$ field theory which is proposed to describe the space of branes with fixed charge involves a superpotential with relations among the vacuum equations, and relations among these relations ...

A. Full supersymmetry transformations and lagrangian

In this appendix we give a more detailed description of the supersymmetry structure of our models, and present the model for a sequence with an arbitrary number of nodes.

First we present the gauge-covariantized supersymmetry transformations for the two-step case and show that we get all of the terms we need in the action to get the desired massless degrees of freedom. (We do not write down terms in the bulk action which are given in [5].) We suppress indices on the fields.

The transformations of the bulk multiplets under the reduced superalgebra are:

$$\begin{aligned} [Q,\phi] &= -i\Theta & \qquad [Q\dagger,\phi] &= 0 \\ \{Q,\Theta\} &= 0 & \qquad \{Q\dagger,\Theta\} &= 2\nabla_0\phi \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \{Q,\eta\} &= F & \qquad \{Q\dagger,\eta\} &= 2\tilde{\nabla}_1\phi\dagger \\ [Q,F] &= 0 & \qquad [Q\dagger,F] &= -2i\nabla_0\eta + 2i\tilde{\nabla}_1\Theta + 2iq\phi\left(\lambda_+\dagger + \lambda_-\dagger\right), \end{aligned} \tag{A.2}$$

where $\tilde{\nabla}_1 \equiv \frac{i}{2\sqrt{2}}\{Q, Q_+\dagger + Q_-\dagger\}$, the anticommutator of Q with the broken supercharge.

We will introduce the notation ξ for the fermionic superpartner of $\wp$. So:

$$
\begin{aligned}
\{Q,\beta'^{\dagger}\} &= b' & \{Q,\beta'\} &= 0\\
\{Q\dagger,\beta'^{\dagger}\} &= 0 & \{Q\dagger,\beta'\} &= b'^{\dagger}\\
[Q,b'] &= 0 & [Q,b'^{\dagger}] &= -2i\nabla_0\beta'\\
[Q\dagger,b'] &= -2i\nabla_0\beta'^{\dagger} & [Q\dagger,b'^{\dagger}] &= 0\\
[Q,\wp] &= -i\xi & [Q,\wp\dagger] &= +id_2\dagger\beta'\\
[Q\dagger,\wp] &= +i\beta'^{\dagger}d_2 & [Q\dagger,\wp\dagger] &= -i\xi\dagger\\
\{Q,\xi\} &= 0 & \{Q,\xi\dagger\} &= 2\nabla_0\wp\dagger + d_2\dagger b'^{\dagger} - i(d_{2,a}\dagger\theta^{a\dagger})\beta'\\
\{Q\dagger,\xi\} &= 2\nabla_0\wp + b'd_2 + i\beta'^{\dagger}(d_{2,a}\theta^a) & \{Q\dagger,\xi\dagger\} &= 0\,.
\end{aligned}
\tag{A.3}
$$

So the $d^2\theta$ integral of $\beta'^{\dagger}\beta'$ is

$$
b'b'^{\dagger} - i(\nabla_0\beta'^{\dagger})\beta' + i\beta'^{\dagger}(\nabla_0\beta')
$$

and the $d^2\theta$ integral of $\wp\wp\dagger$ contains five types of term:

- a 'target space magnetic field' term

$$
+i(\nabla_0\wp)\wp\dagger - i\wp(\nabla_0\wp\dagger)\,; \tag{A.4}
$$

- quadratic terms for the gauge-invariant ξ fermions:

$$
-\xi\xi\dagger\,; \tag{A.5}
$$

- mass terms for the β' fermions:

$$
+\beta'^{\dagger}d_2d_2\dagger\beta'\,; \tag{A.6}
$$

- some F-terms giving rise to a bosonic potential for $\wp$:

$$
ib'd_2\wp\dagger - i\wp d\dagger_2 b'^{\dagger}\,; \tag{A.7}
$$

- and some cross terms between β' and $\wp$ which have no obvious role:

$$
-\beta'^{\dagger}(d_{2,a}\theta^a)\wp\dagger - \wp(d\dagger_{2,a}\theta^{a\dagger})\beta'\,; \tag{A.8}
$$

- we also have the usual kinetic terms for the β-fermions:

$$
\int d^2\theta\beta\dagger\beta = +b\dagger b - i(\nabla_0\beta\dagger)\beta + i\beta\dagger(\nabla_0\beta)\,. \tag{A.9}
$$

Also, the superpotential contributes the following component terms:

$$
\begin{aligned}
\int d\theta W &\equiv i\int d\theta\wp d_1\beta = i\int d\theta\wp d_1\beta\\
&= \xi d_1\beta + \beta\dagger d\dagger_1\xi\dagger + \wp(d_{1,a}\theta^a)\beta + \beta\dagger(d\dagger_{1,a}\theta^{a\dagger})\wp\dagger + i\wp d_1 b - ib\dagger d\dagger_1\wp\dagger
\end{aligned}
\tag{A.10}
$$

The model with many nodes. For an arbitrary number of steps, the action is

$$\int d^2\theta \sum_n \left(\beta\dagger_{(n)}\beta_{(n)} + \wp_{(n+1)}\wp_{(n+1)}\dagger\right) + \int d\theta\, W + \text{h.c.}$$

and the on-shell supersymmetry transformations are

$$\begin{aligned}
\{Q\dagger,\ \beta_{(1)}\} &= 0 \qquad \{Q, \beta_{(1)}\} = -id_1\dagger \cdot \wp_{(2)}\dagger \\
&\cdots \\
\{Q\dagger,\ \beta_{(n)}\} &= id_{n-1} \cdot \wp_{(n-1)}\dagger \qquad \{Q,\ \beta_{(n)}\} = -id_n\dagger \cdot \wp_{(n+1)}\dagger \\
[Q\dagger,\ \wp\dagger_{(n+1)}] &= id_n \cdot \beta_{(n)} \qquad [Q,\ \wp\dagger_{(n+1)}] = id_{n+1}\dagger \cdot \beta_{(n+2)} \\
&\cdots
\end{aligned} \tag{A.11}$$

B. Formulation with shift symmetries

In the body of this paper we have given a presentation of our models which clearly exhibits the spectrum of fields and their interactions. This presentation has the drawback that the supersymmetry transformations of the fields are complicated and the closure of the supersymmetry algebra is not manifest. In this appendix we present a formulation of our models in which the supersymmetry transformations of all fields are simple but which involves a number of nonlinearly-realized gauge symmetries. The formulation of the models given in the body of the paper results from fixing these gauge symmetries. This appendix clarifies the relation between the modified chiral constraints and the "fermionic gauge symmetries" of [29, 30], and is how our models were initially constructed.

We again present the construction for a two-step sequence, 3.2. Introduce a fermi multiplet β_1 of bulk gauge charge n_1, a chiral multiplet $\tilde{\wp}_2$ of bulk gauge charge $-n_2$, and an unconstrained multiplet V_3 of gauge charge n_3 whose lowest component is a boson. The usual superpotential

$$W = \tilde{\wp}_2 d_1 \beta$$

respects the following shift symmetry, which we gauge:

$$\begin{aligned}
\tilde{\wp}_1 &\mapsto \tilde{\wp}_1 + \Omega_2 d_2 \\
V_3 &\mapsto V_3 + \Omega_2\,.
\end{aligned} \tag{B.1}$$

The gauge parameter Ω_2 is a chiral multiplet, $Q\dagger\Omega_2 = 0$. Then the shift-symmetry invariant field $\wp_2 \equiv \tilde{\wp}_2 - V_3 d_2$ satisifes the constraint $Q\dagger\wp_2 = \beta_3\dagger d_2$ with $\beta_3\dagger \equiv -Q\dagger V_3$. $\beta_3\dagger$ is gauge invariant and satisfies $Q\dagger\beta_3\dagger = 0$.

In order to extend the sequence by another step, enlarge the gauge symmetry. Relax the condition $Q^\dagger\Omega_2 = 0$ on the gauge parameters and demand merely that

$Q^\dagger\Omega_2 = \Omega_3 d_3$ for some Ω_3 with $Q^\dagger\Omega_3 = 0$. Note again that the fact that the maps d_l form a complex is crucial for closure of the superalgebra. Under this modified shift symmetry, the quantity $\wp_2$ is still gauge invariant, but $\tilde{\beta}_3\dagger \equiv -Q^\dagger V_3$ is not. Introduce a new gauge (i.e. unconstrained) multiplet V_4 (whose lowest component is a fermion) and assign it a transformation $\Sigma_4 \mapsto \Sigma_4 + \Omega_3$. Then $\beta_3^\dagger \equiv \tilde{\beta_3}^\dagger - \Sigma_4 d_3$ is gauge invariant, and satisfies the deformed chiral constraint $Q^\dagger\beta_3^\dagger = \wp_4 d_3$ where $\wp_4 \equiv -Q^\dagger\Sigma_4$ is gauge invariant and annihilated by $Q^\dagger$. The generalization of this formulation to an arbitrary number of steps should be clear.

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